

Modelling comonotonic group-life under dependent decrement causes

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Abstract

Comonotonicity had been a extreme case of dependency between random variables. This article consider an extension of single life model under multiple dependent decrement causes to the case of comonotonic group-life.

1 Introduction

The concept of comonotonicity was proposed for research of economics, and it had been introduced into the insurance actuarial in 1990's (see. Dhaene J. *et al.*[3]) for ordering risks and stop-loss premium determination (see. Dhaene J. *et al.* [4]). Research results show that comonotonicity had been a extreme case of dependency between random variables (see. Dhaene J. *et al.*[3],[4],[5], Ribas C.[8]). Thus, for example, comonotonicity between lives in a group implies a special characteristics of the group-life that one member of the group attains venerable age then all of other members of the group also attain venerable age. The decrement models with respect to multiple dependent causes of withdrawal have been received a attention from life insurance area, Carriere JF.(1994,1998) [1],[2] and Vladimir K. *et al.*(2007)[9] consider the survival of the single life under multiple dependent decrement causes, their results are mainly based on

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the copula functions. A statistical estimation on the net survival functions using the observations of crude survivals was proposed. Wang *et al.*(2009) [10] has considered the decrement model for comonotonic group-life with status of joint-survival and last survival under independent multiple decrement causes. The distribution model with statistical analysis on comonotonic group-life under multiple dependent decrement causes will be considered in the present paper. The comonotonicity properties seemingly make that each group-life can be represented by any single member of the group, which would leads to a simple way inducing a joint distribution for the considered collection of comonotonic group-life. This paper is organized as follows. In Section 2, the main concepts like comonotonicity, copula functions, survival function as well as some useful obtained results are mentioned. in Section 3, we construct a model for dependent multiple comonotonic group-life.

2 Preliminaries

Definition 2.1 (Dhaene J.[3]) Let R^n be a n -dimensional Euclidean space, $A \subset R^n$ is said to be a comonotonic vectors set if for any vectors $\bar{x}, \bar{y} \in A$, it holds that $\bar{x} \leq \bar{y}$ or $\bar{x} \geq \bar{y}$. Where $\bar{x} \leq (\geq) \bar{y}$ means that $x_i \leq (\geq) y_i, i = 1, \dots, n$.

Following [3], we see that the earlier concept of two comonotonic random variables due to Schmeidler (1986), Yarri(1987) 's work on economics. Let (X, Y) be a two dimensional random vector on the probability space (Ω, \mathcal{F}, P) . If for any $\omega_1, \omega_2 \in \Omega$ there always holds that $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$, then, (X, Y) is said to be comonotonic, this definition was weaken as $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0, P.a.s.$ around 1997. For a n -dimensional random vector $\bar{X} = (X_1, \dots, X_n)$ defined on (Ω, \mathcal{F}, P) , if there exists a subset $A \subset R^n$ such that $P(\bar{X} \in A) = 1$, then A is said to be a support of \bar{X} . A Random vector $\bar{X} = (X_1, \dots, X_n)$ is said to be comonotonic if its support is comonotonic.

Lemma 2.1(Dhaene J. *et al.*[3]) A random vector $\bar{X} = (X_1, \dots, X_n)$ on probability space (Ω, \mathcal{F}, P) is comonotonic if and only if the following equivalent conditions hold:

- (1) \bar{X} has a comonotonic support;
- (2) For all values of \bar{X} , $\bar{x} = (x_1, \dots, x_n)$, the joint distribution of \bar{X} is

$$F_{\bar{X}}(\bar{x}) = \min\{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\};$$

(3) Let U be a random variable uniformly distributed on interval $(0,1)$. Then,

$$\overline{X} =^d (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \dots, F_{X_n}^{-1}(U));$$

(4) There exists random variable z and non-decreasing real valued functions $f_i (i = 1, \dots, n)$ such that $\overline{X} =^d (f_1(z), f_2(z), \dots, f_n(z))$.

Following the example of [1],[2],[9], the dependent multiple decrements models for single life in life insurance have been investigated extensively. Here we only introduce some basic notation and main proposed methods, for other details the readers are referred to [1],[2],[9] and the literatures therein.

We consider a group of lives, exposed to m competing causes of withdrawal from the group. It is assumed that each individual may withdraw from any single one of the m causes. It is assumed that, at birth, each individual is assigned a vector of times T_1, \dots, T_m , $0 \leq T_j < \infty$, $j = 1, \dots, m$, representing individual's potential lifetime, if the individual were to withdraw from each one of the m causes. Obviously, the actual lifetime span is the minimum of all the T_1, \dots, T_m . Thus, it is clear that under this model the lifetimes T_1, \dots, T_m are unobservable, and we can only observe the $\min(T_1, \dots, T_m)$. In the classical multiple-decrement theory the random variables T_1, \dots, T_m are assumed independent, however, decrement in many real-life actuarial applications tend to be dependent, and the random variables T_1, \dots, T_m are considered stochastically dependent and also non-defective, i.e. $P(T_j < \infty) = 1$. The joint distribution

$$F(t_1, \dots, t_m) = P(T_1 \leq t_1, \dots, T_m \leq t_m),$$

the multivariate joint survival function

$$S(t_1, \dots, t_m) = P(T_1 > t_1, \dots, T_m > t_m)$$

which is considered absolutely continuous and where $t_j \geq 0$, for $j = 1, \dots, m$. The overall survival function of an individual aged $x \geq 0$ is defined through random variable $\min(T_1, \dots, T_m)$ as

$$\mathbb{S}(t) := S(t, \dots, t) = P(T_1 > t, \dots, T_m > t) = P(\min(T_1, \dots, T_m) > t).$$

The crude survival function $S^{(j)}(t)$ is defined as

$$S^{(j)}(t) = P(\min(T_1, \dots, T_m) > t, \min(T_1, \dots, T_m) = T_j).$$

The net survival function $S'^{(j)}(t)$ is defined as $S'^{(j)}(t) = P(T_j > t)$. Note that $S'^{(j)}(t)$ is the marginal survival function, due to cause alone, associated with the joint multivariate survival function

$$S(t_1, \dots, t_m) = P(T_1 > t_1, \dots, T_m > t_m).$$

Thus, we can view $F(t_1, \dots, t_m) = P(T_1 \leq t_1, \dots, T_m \leq t_m)$ as a multivariate distribution with marginal distributions $F'^{(j)}(t) = 1 - S'^{(j)}(t), j = 1, \dots, m$. If we know $S'^{(j)}(t)$, we can identify and calculate the joint survival function $S(t_1, \dots, t_m)$, and hence evaluate the overall survival function $S(t, \dots, t)$ under some assumption of copula functions.

The copula is one of the most useful tools for handling multivariate distributions with given univariate marginals F_1, \dots, F_m . Formally, a copula C is a cumulative distribution function, defined on $[0, 1]^m$, with uniform marginals. Given a copula C , if one defines

$$F(x_1, \dots, x_m) = C(F_1(x_1), F_2(x_2), \dots, F_m(x_m)), (x_1, \dots, x_m) \in R^m,$$

then F is a multivariate distribution with univariate marginals F_1, \dots, F_m . According to Sklar theorem (1959)[9], Given a continuous joint distribution function $F(x_1, \dots, x_m)$ of m -dimensional random vector $\overline{X} = (X_1, \dots, X_m)$, with marginal distribution functions

$$F_1(x_1), F_2(x_2), \dots, F_m(x_m), (x_1, \dots, x_m) \in R^m,$$

there corresponds to it a unique m -dimensional Copula function C can be constructed as

$$C(u_1, \dots, u_m) = F(F_1^{-1}(u_1), \dots, F_m^{-1}(u_m)), (u_1, \dots, u_m) \in [0, 1]^m.$$

Note that different multivariate distributions F may have the same copula. Most of the multivariate dependence structure properties of F are in the copula function, which is independent of the marginals and which is, in general, easier to handle than the original F .

The Sklar theorem can be restated to express the multivariate survival function $S(x_1, \dots, x_n)$ via an appropriate copula \overline{C} called the survival copula of (X_1, \dots, X_n) . Thus,

$$S(x_1, \dots, x_n) = \overline{C}(S_1(x_1), \dots, S_n(x_n)).$$

Where $S_i(x_i) = 1 - F_i(x_i) = 1 - u_i, i = 1, \dots, n$. When $n = 2$, we have $\overline{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$. There are many copula functions constructed by using the given distributions, e.g. the multivariate Gaussian copula, Student's t -copula, and the popular Archimedean copulas constructed by $C^A(u_1, u_2) = \phi^{-1}(\phi(u_1) + \phi(u_2))$, where ϕ is a continuous, convex function called generator, such that $\phi(1) = 0, \phi(0) = +\infty$, of which the frequently used Clayton copula, Gumbel copula, etc. (see. Nelsen (1999)[7]).

Some decrement models for independent as well as comonotonic group-life with specified status under independent causes of decrements have been obtained by Wang (see. Wang Z.J. and Wang D. (2009)[10]).

3 Model for comonotonic group-life under dependent decrement causes

Let x_1, \dots, x_n be a group of n lives with comonotonic structure, and each member x_i of the group can be withdrawn with one of m different dependent causes of decrements T_1, \dots, T_m . We start with the simple case where the status of the lives group are determined as $g(x_1, \dots, x_n)$, e.g., which can be the joint-survival status $(x_1 \cdots x_n)$, or the last-survival status $(\overline{x_1 \cdots x_n})$, etc.. $g(x_1, \dots, x_n)$ can be withdrawn with anyone of m different dependent causes of decrements.

$$T_1(g(x_1, \dots, x_n)), \dots, T_m(g(x_1, \dots, x_n))$$

are the latent decrement times of any status $g(x_1, \dots, x_n)$ of the group of n lives. We can only observe the time

$$T(g(x_1, \dots, x_n)) = \min\{T_1(g(x_1, \dots, x_n)), \dots, T_m(g(x_1, \dots, x_n))\}.$$

Since the dependency of the decrements causes, the latent decrement times

$$T_1(g(x_1, \dots, x_n)), \dots, T_m(g(x_1, \dots, x_n))$$

are dependent. Assume that $T_1(g(x_1, \dots, x_n)), \dots, T_m(g(x_1, \dots, x_n))$ has joint distribution function and marginals as

$$F_{T_1, T_2, \dots, T_m}(t_1, t_2, \dots, t_m) = P(T_1(g(x_1, \dots, x_n)) \leq t_1, \dots, T_m(g(x_1, \dots, x_n)) \leq t_m);$$

$$F_{T_1}(t_1) = P(T_1(g(x_1, \dots, x_n)) \leq t_1), \dots, F_{T_m}(t_m) = P(T_m(g(x_1, \dots, x_n)) \leq t_m),$$

respectively. According to the Sklar theorem, there exists a copula function C such that

$$F_{T_1, T_2, \dots, T_m}(t_1, t_2, \dots, t_m) = C(F_{T_1}(t_1), \dots, F_{T_m}(t_m)).$$

Example 1 Consider the case where $g(x, y) = (xy)$ and $g(x, y) = (\overline{xy})$, and lives $(x), (y)$ are comonotonic and $m = 2$.

(1) Let $F_{T_1, T_2}(t_1, t_2)$ be the joint distribution function of $T_1(xy), T_2(xy)$. Let $F_{T_1}(t_1), F_{T_2}(t_2)$ be the distribution functions of $T_1(xy), T_2(xy)$ respectively. Then, there exists a Clayton copula function C such that

$$F_{T_1, T_2}(t_1, t_2) = \left[(F_{T_1}(t_1))^{-\theta} + (F_{T_2}(t_2))^{-\theta} - 1 \right]^{-\frac{1}{\theta}},$$

where the parameter $\theta > 0$, and

$$F_{T_1}(t_1) = \max\{ {}_{t_1}q_x^{(1)}, {}_{t_1}q_y^{(1)} \}, F_{T_2}(t_2) = \max\{ {}_{t_2}q_x^{(2)}, {}_{t_2}q_y^{(2)} \}$$

by the lemma 2.1 (3). Where ${}_tq_x^{(i)} := P(T_i(x) \leq t)$ and so are in the sequel.

(2) The conditions of (1) but $g(x, y) = (\overline{xy})$. Then, there exists Gumbel copula function C such that

$$F_{T_1, T_2}(t_1, t_2) = \exp \left\{ - \left[(-\ln(F_{T_1}(t_1)))^\theta + (-\ln(F_{T_2}(t_2)))^\theta \right]^{\frac{1}{\theta}} \right\},$$

where the parameter $\theta > 0$ can be determined by the given values of the Kandel statistics $\tau_\theta = 1 - \frac{1}{\theta}$, and

$$F_{T_1}(t_1) = \min\{ {}_{t_1}q_x^{(1)}, {}_{t_1}q_y^{(1)} \}, F_{T_2}(t_2) = \min\{ {}_{t_2}q_x^{(2)}, {}_{t_2}q_y^{(2)} \}$$

by the lemma 2.1 (3).

(3) For the observable decrement time $T(g(x, y))$, $g(x, y) = (xy)$, we have

$$\begin{aligned} {}_tq_{xy}^{(\tau)} &= P(T(xy) \leq t, j = 1, 2) \\ &= P(\min\{T_1(xy), T_2(xy)\} \leq t) \\ &= 1 - \overline{C}\left(1 - \max\{{}_tq_x^{(1)}, {}_tq_y^{(1)}\}, 1 - \max\{{}_tq_x^{(2)}, {}_tq_y^{(2)}\}\right), \end{aligned}$$

where \overline{C} is a survival copula function which can be transformed into Gumbel or Clayton form. Similarly, we have

$${}_tq_{\overline{xy}}^{(\tau)} = 1 - \overline{C}\left(1 - \min\{{}_tq_x^{(1)}, {}_tq_y^{(1)}\}, 1 - \min\{{}_tq_x^{(2)}, {}_tq_y^{(2)}\}\right)$$

for status $g(x, y) = (\overline{xy})$.

Note that here the decrement probabilities ${}_tq_x^{(1)}, {}_tq_y^{(1)}, {}_tq_x^{(2)}, {}_tq_y^{(2)}$, in fact, are

the net decrement probabilities corresponding to the net survival functions $S^{(1)}(t), S^{(2)}(t)$, we could not obtain them by the intuitive observations. According to Carriere JF. (1994)[1] and V.K. Kaishev *et al.* (2007)[9], using a nonlinear system of differential equations which represent the functional relation between the crude survival function $S^{(j)}(t)$ and the net survival function $S'^{(j)}(t)$, under condition that $S^{(j)}(t)$ can be estimated based on intuitive observations, we can solve and estimate $S'^{(j)}(t)$. Therefore, we can obtain the estimates of ${}_{tq_x}^{(1)}, {}_{tq_y}^{(1)}, {}_{tq_x}^{(2)}, {}_{tq_y}^{(2)}$.

Consider a general case where the status of the group of n lives are not specified, and the group of n lives is comonotonic and each life of the group exposed to m dependent decrements causes. Then, we have m dependent comonotonic future lifetime vectors as $\bar{T}_i := (T_i(x_1), \dots, T_i(x_n)), i = 1, \dots, m$. In fact, the conclusion that there are dependent relations between the vectors $\bar{T}_1, \dots, \bar{T}_m$ can be easily verified by that the covariance matrix $COV(\bar{T}_k, \bar{T}_j) \neq O, k \neq j$ (since $Cov(T_i(x_l), T_j(x_l)) \neq 0$.) Note that this dependency in general can be modeled by the multivariate distribution function of the vector $(\bar{T}_1, \dots, \bar{T}_m)$ which has a dimension nm , we also note that each component vector \bar{T}_i has a joint distribution $F_{\bar{T}_i}(\bar{t}_i)$ which equals to a comonotonic copula $\min\{F_{T_i(x_1)}(t_{i1}), \dots, F_{T_i(x_n)}(t_{in})\}$ by Lemma 2.1 (2), where $\bar{t}_i := (t_{i1}, \dots, t_{in})$. It is desirable to construct the multivariate distribution function F of the vector $(\bar{T}_1, \dots, \bar{T}_m)$ with the multivariate marginals $F_{\bar{T}_1}, \dots, F_{\bar{T}_m}$, the comonotonic copulas $\min\{F_{T_i(x_1)}(t_{i1}), \dots, F_{T_i(x_n)}(t_{in})\}, i = 1, \dots, m$. According to H.Lin *et al.*(1996) only the copula of independent case can be considered for modeling the joint multivariate distribution function with the distributions of multivariate marginals, i.e. if

$$F_{\bar{T}_1, \dots, \bar{T}_m} = C(F_{\bar{T}_1}, \dots, F_{\bar{T}_m}).$$

then the copula $C(u_1, \dots, u_m) = \prod_{i=1}^m u_i$. However, this is not our case. The comonotonicity makes things much easier than above case.

Theorem 3.1 *Let x_1, \dots, x_n be a comonotonic group-life exposed to m dependent decrement causes T_1, \dots, T_m . Assume that for one member x_l of the group the future lifetime vector $(T_1(x_l), \dots, T_m(x_l))$ has a joint distribution $F_{T_1(x_l), \dots, T_m(x_l)}$ with marginals $F_{T_1(x_l)}, \dots, F_{T_m(x_l)}$. Then, their copula C derived from Sklar theorem is a copula for modelling the dependency in comonotonic future lifetime vectors $\bar{T}_1, \dots, \bar{T}_m$.*

Proof By Sklar theorem there exists a copula function C such that

$$F_{T_1(x_l), \dots, T_m(x_l)}(t_{1l}, \dots, t_{ml}) = C(F_{T_1(x_l)}(t_{1l}), \dots, F_{T_m(x_l)}(t_{ml})),$$

and $F_{T_i(x_l)}(t_{il}) \geq \min\{F_{T_i(x_1)}(t_{i1}), \dots, F_{T_i(x_n)}(t_{in})\} = F_{\bar{T}_i}(\bar{t}_i), i = 1, \dots, m$, since copula C is a distribution function, thus we have

$$C(F_{T_1(x_l)}(t_{1l}), \dots, F_{T_m(x_l)}(t_{ml})) \geq C(F_{\bar{T}_1}(\bar{t}_1), \dots, F_{\bar{T}_m}(\bar{t}_m)),$$

the most dependency structure properties of the vectors $\bar{T}_1, \dots, \bar{T}_m$ are in the later function $C(F_{\bar{T}_1}(\bar{t}_1), \dots, F_{\bar{T}_m}(\bar{t}_m))$, which can be defined as a joint distribution function

$$F(t_{11}, \dots, t_{1n}, t_{21}, \dots, t_{2n}, \dots, t_{m1}, \dots, t_{mn})$$

of mn -dimension. \square

Corollary 3.1 Assume the condition of Theorem 3.1. Then, the copula C is also a survival copula for the joint vector survival function $S_{\bar{T}_1, \dots, \bar{T}_m}(\bar{t}_1, \dots, \bar{t}_m)$, i.e.

$$S_{\bar{T}_1, \dots, \bar{T}_m}(\bar{t}_1, \dots, \bar{t}_m) = C(S_{\bar{T}_1}(\bar{t}_1), \dots, S_{\bar{T}_m}(\bar{t}_m)).$$

Proof By Lemma 2.1 (3), we have

$$\begin{aligned} S_{\bar{T}_i}(\bar{t}_i) &= P(T_i(x_1) > t_{i1}, \dots, T_i(x_n) > t_{in}) = P(F_{T_i(x_1)}^{-1}(U) > t_{i1}, \dots, F_{T_i(x_n)}^{-1}(U) > t_{in}) \\ &= P(U > F_{T_i(x_1)}(t_{i1}), \dots, U > F_{T_i(x_n)}(t_{in})) = \max\{F_{T_i(x_1)}(t_{i1}), \dots, F_{T_i(x_n)}(t_{in})\} \\ &\geq F_{T_i(x_l)}(t_{il}), i = 1, \dots, m. \end{aligned}$$

Thus,

$$C(F_{T_1(x_l)}(t_{1l}), \dots, F_{T_m(x_l)}(t_{ml})) \leq C(S_{\bar{T}_1}(\bar{t}_1), \dots, S_{\bar{T}_m}(\bar{t}_m)),$$

the later copula can be viewed as a survival copula and we define it as the joint vector survival function $S_{\bar{T}_1, \dots, \bar{T}_m}(\bar{t}_1, \dots, \bar{t}_m)$. \square

The overall survival, crude survival for the future life time vectors can be similarly defined. It is important to note that $\min\{\bar{T}_1, \dots, \bar{T}_m\}$ is the observable decrement time of whole group-life. The crude survival $S^{(j)}(\bar{t}) := P(\min\{\bar{T}_1, \dots, \bar{T}_m\} > \bar{t}, \min\{\bar{T}_1, \dots, \bar{T}_m\} = \bar{T}_j)$. If we view the joint vector survival function $S_{\bar{T}_1, \dots, \bar{T}_m}(\bar{t}_1, \dots, \bar{t}_m)$ as a mn -dimensional joint survival function

$$S_{T_1(x_1), \dots, T_1(x_n), \dots, T_m(x_1), \dots, T_m(x_n)}(t_{11}, \dots, t_{1n}, \dots, t_{m1}, \dots, t_{mn}),$$

Then, the crude survival function for the case where x_l is selected as a representative of the group can be defined as $S^{(j)}(t) = P(\min\{T_1(x_1), \dots, T_1(x_n), \dots, T_m(x_1), \dots, T_m(x_n)\} > t,$

$$\min\{T_1(x_1), \dots, T_1(x_n), \dots, T_m(x_1), \dots, T_m(x_n)\} = T_j(x_l)).$$

Theorem 3.2(1) *If the copula C in theorem 3.1 and corollary 3.1 is differentiable with respect to j th variable and $S_{\overline{T}_j}(\overline{t}_j)$ is partially differentiable with respect to $t_{jl} > 0$ for all $j = 1, \dots, m$, then*

$$\frac{d}{dt}S^{(j)}(t) = C_j(S_{\overline{T}_1}(\overline{t}_1), \dots, S_{\overline{T}_m}(\overline{t}_m)) \times \frac{\partial(S_{\overline{T}_j}(\overline{t}_j))}{\partial t_{jl}} \Bigg|_{t_{jl}=t}.$$

where $C_j(u_1, \dots, u_m) = \frac{\partial}{\partial u_j}C(u_1, \dots, u_m)$.

(2) *If the copula C is n -order differentiable with respect to its every variables and each survival function $S_{\overline{T}_j}(\overline{t}_j)$ is n -order partially differentiable with respect to its every variables, then*

$$\frac{\partial^{(n)}}{\partial t_1 \partial t_2 \dots \partial t_n} S^{(j)}(\overline{t}) = \frac{\partial^{(n)}}{\partial t_{j1} \partial t_{j2} \dots \partial t_{jn}} C(S_{\overline{T}_1}(\overline{t}_1), \dots, S_{\overline{T}_m}(\overline{t}_m)) \Bigg|_{(t_{j1}, \dots, t_{jn})=(t_1, \dots, t_n)},$$

especially when $n = 2$, it holds

$$\frac{\partial^2 S^{(j)}(t_1, t_2)}{\partial t_1 \partial t_2} = \frac{\partial S_{\overline{T}_j}}{\partial t_{j1}} \frac{\partial^2 C}{\partial S_{\overline{T}_j}^2} \frac{\partial S_{\overline{T}_j}}{\partial t_{j2}} \Bigg|_{(t_{j1}, t_{j2})=(t_1, t_2)} + \frac{\partial C}{\partial S_{\overline{T}_j}} \frac{\partial^2 S_{\overline{T}_j}}{\partial t_{j1} \partial t_{j2}} \Bigg|_{(t_{j1}, t_{j2})=(t_1, t_2)},$$

$j = 1, \dots, m$.

Proof (1) See the proof of theorem 6 and lemma 1 in Carriere JF.[1](1994).

(2) Since

$$\begin{aligned} S^{(j)}(\overline{t}) &= P(\min\{\overline{T}_1, \dots, \overline{T}_m\} > \overline{t}, \min\{\overline{T}_1, \dots, \overline{T}_m\} = \overline{T}_j) \\ &= P(\overline{T}_j > \overline{t}_j, \overline{T}_k > \overline{T}_j (k \neq j)) = \int_{t_1}^{\infty} \dots \int_{t_n}^{\infty} \left\{ \int_{t_{j1}}^{\infty} \dots \int_{t_{jn}}^{\infty} \dots \int_{t_{j1}}^{\infty} \dots \int_{t_{jn}}^{\infty} \right. \\ &\quad \left. f(t_{11}, \dots, t_{1n}, \dots, t_{m1}, \dots, t_{mn}) \prod_{k \neq j} dt_{k1} \dots dt_{kn} \right\} dt_{j1} \dots dt_{jn} \\ &= \int_{t_1}^{\infty} \dots \int_{t_n}^{\infty} \left\{ \frac{\partial^{(n)}}{\partial t_{j1} \partial t_{j2} \dots \partial t_{jn}} S_{T_1(x_1), \dots, T_1(x_n), \dots, T_m(x_1), \dots, T_m(x_n)} \right. \\ &\quad \left. (t_{11}, \dots, t_{1n}, \dots, t_{m1}, \dots, t_{mn}) \Bigg|_{(t_{k1}, \dots, t_{kn})=(t_{j1}, \dots, t_{jn}), \forall k} \right\} dt_{j1} \dots dt_{jn} \\ &= \int_{t_1}^{\infty} \dots \int_{t_n}^{\infty} \left\{ \frac{\partial^{(n)}}{\partial t_{j1} \partial t_{j2} \dots \partial t_{jn}} C(S_{\overline{T}_1}(\overline{t}_1), \dots, S_{\overline{T}_m}(\overline{t}_m)) \Bigg|_{(t_{k1}, \dots, t_{kn})=(t_{j1}, \dots, t_{jn}), \forall k} \right\} dt_{j1} \dots dt_{jn}. \end{aligned}$$

Thus,

$$\frac{\partial^{(n)}}{\partial t_1 \partial t_2 \dots \partial t_n} S^{(j)}(\overline{t}) = \frac{\partial^{(n)}}{\partial t_{j1} \partial t_{j2} \dots \partial t_{jn}} C(S_{\overline{T}_1}(\overline{t}_1), \dots, S_{\overline{T}_m}(\overline{t}_m)) \Bigg|_{(t_{j1}, \dots, t_{jn})=(t_1, \dots, t_n)}.$$

□

Theorem 3.2 indicates that in the case of comonotonic group-life, the relationship between crude survivals and net survivals is formulated by some much complicated n -order non-linear partial differential equations, which in most case is unsolvable when we have some knowledge of the crude survivals. When $n = 2$, with the conclusion (2) of theorem 3.2, If we have obtained the copula function C through considering some member of the group, then by this fixed copula C , we may use some numerical methods introduced in Carriere JF.[1] and Vladimir K Kaishev *et al.*[9] to obtain the estimates of $S_{\overline{T}_1}(\bar{t}_1), \dots, S_{\overline{T}_m}(\bar{t}_m)$ based on the observations of the crude survivals $S^{(j)}(\bar{t}), j = 1, \dots, m$. Then the estimation of joint survival $S_{\overline{T}_1, \dots, \overline{T}_m}(\bar{t}_1, \dots, \bar{t}_m)$ can be carried out.

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